Project systems theory

Final exam 2018–2019, Thursday 24 January 2019, 9:00 – 12:00

Problem 1

(2 + 4 + 9 = 15 points)

Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_1x_2 \\ -x_1 + x_1^2 + x_2^2 + u \end{bmatrix}, \qquad y = 2x_1x_2 \tag{1}$$

with state $x = [\, x_1 \;\; x_2 \,]^{\mathrm{T}}$, input $u \in \mathbb{R}$, and output $y \in \mathbb{R}$.

(a) Choose $u(t) = 0$ for all $t \geq 0$. Show that

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2}$$

is an equilibrium point of the nonlinear system (1).

(b) Take again $u(t) = 0$ for all $t \geq 0$. Determine all equilibrium points of the nonlinear system (1), i.e., compute the equilibrium points different from (2).

(c) Linearize the nonlinear system (1) around the equilibrium point (2).

Problem 2

(20 points)

Consider the family of polynomials

$$\mathcal{P}(\lambda) = \left\{ \lambda^3 + \theta_2\lambda^2 + a\lambda + \theta_0 \,\middle|\, 2a \leq \theta_2 \leq 3a,\ a \leq \theta_0 \leq 4a \right\} \tag{3}$$

with a a real number. Determine for which values of a the family $P(\lambda)$ is stable.

Hint. Recall that the family $P(\lambda)$ is said to be stable if each polynomial belonging to $P(\lambda)$ is stable.

Problem 3

(6 + 4 = 10 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -2 & 8 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} x(t). \tag{4}$$

(a) Determine the unobservable subspace $\mathcal{N}$ and give a basis for this subspace.

(b) Let $x(0) = x_0 \in \mathcal{N}$ be an initial condition in the unobservable subspace $\mathcal{N}$ and take $u(t) = 0$ for all $t \geq 0$. Give the corresponding output trajectory $y(\cdot)$ of the system.

Problem 4 (4 + 4 + 4 + 12 + 6 = 30 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5}$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & 1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \tag{6}$$

(a) Is the system (asymptotically) stable?

(b) Is the system controllable?

(c) Is the system stabilizable?

In the remainder of this problem, the objective is to find a stabilizing state feedback for (5)–(6).

(d) Find a nonsingular matrix T such that

$$T^{-1}AT = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -a_2 & -a_1 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

Hint. Following the block-diagonal structure of A in (6), choose T such that it has a block-diagonal structure as well.

(e) Use the matrix T from problem (d) to obtain a state feedback of the form $u = Fx$ with

$$F = \begin{bmatrix} 0 & f_2 & f_1 \end{bmatrix}, \tag{8}$$

such that the closed-loop system matrix $A + BF$ has eigenvalues at -3, -1, and -1.

Problem 5 (15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{9}$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$. Let $m = 1$. Show that the system cannot be controllable if there are two linearly independent eigenvectors of A^{T} corresponding to the same eigenvalue λ.

(10 points free)